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SMALL DEFORMATIONS OF A NONLINEAR VISCOELASTIC TUBE: THEORY AND APPLICATION TO POLYPROPYLENE

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CONTENTS

The response of an isotropic, nonlinear viscoelastic, thin-walled tube to combinations of axial force F, axial couple G and pressure difference p is considered theoretically and experimentally. Theory is based on the membrane theory of thin shells, applied to a thin-walled circular cylindrical tube. The components of two dimensional stress and strain in the wall of the tube are derived, allowing for arbitrarily large deformations; but restriction to small deformations is shown to be necessary if the history of stress is to be controlled at will through F, G and p. For arbitrary choice of F, G and p as functions of time the strain is shown to depend on three stress tensors $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ independent of time, and three scalar functions of time. An expression for the linear strain tensor in terms of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ is obtained which involves four scalar functions ϕ_0, ϕ_1, ϕ_2, ϕ_3. These functions depend on the invariants of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and on the three scalar functions of time. If any one of F, G, p is always zero then $\boldsymbol{R} = \boldsymbol{0}$ and only ϕ_0, ϕ_1, ϕ_2 are required. In the case of proportional loading ($\boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{0}$) only ϕ_0 and ϕ_1 are required and any one of the three strain components can be calculated from the remaining two.

Creep and recovery experiments under simultaneous axial force and couple were conducted on a thin-walled tube of polypropylene at 65.5 °C. Theory was used to calculate the circumferential tensile strain from the measured shear strain and longitudinal tensile strain. For this particular tube ϕ_0 and ϕ_1 were found to be related in a special manner, implying that nonlinearity can be adequately described by allowing the shear creep compliance to change with stress history. By varying separately combinations of the invariants of $\boldsymbol{P}$, ϕ_1 was found to depend on both hydrostatic and deviatoric components of the applied stress.

1. Introduction

The response of a material to biaxial stress is conveniently studied by subjecting thin-walled tubular specimens to combinations of axial force, axial torque and pressure difference across the wall. This technique is finding increasing application to polymers, which generally deform as nonlinear viscoelastic solids. Experimental results to date have been interpreted in terms of formulae which have only restricted application. See for example the cases of polyvinyl chloride (Onaran & Findley 1965*a*, *b*, 1971; Findley & Lai 1967) polymethyl methacrylate (Baev & Malinin 1966; Zaslawsky 1969) polyurethane (Lai & Findley 1969*a*) polycarbonate (Lai & Findley 1969*b*) polyethylene (Ewing, Turner & Williams 1972, 1973) and polypropylene (Buckley & McCrum 1974). Although for some purposes this approach can be adequate, there is a need for a theoretical framework within which to express the deformation of any such tube. Our aims in this paper, therefore, are first to develop a rather general theory for the deformation of the tube when it is composed of an isotropic nonlinear viscoelastic material, and secondly to apply the theory to some experimental results obtained with a tube of polypropylene.

The theoretical approach is based on the membrane theory of thin shells. It is developed in three parts. In § 2 general expressions for stress and strain in a deformed membrane are derived, subject only to the requirements of mass conservation, equilibrium and objectivity of the observer. These are then evaluated in § 3 for the special case of a thin-walled circular cylindrical tube, subjected to an axial force, axial couple and pressure difference. We show that the stress history can be controlled at will by the experimenter only when deformations of the tube are small. In § 4 the tube is taken to consist of an isotropic but otherwise generally nonlinear viscoelastic material. We derive general expressions for the deformation of the tube under these conditions. The expressions are found to take a simple explicit form involving only four independent material response functions.

The usefulness of the theory is then demonstrated by applying it to creep and recovery experiments under combined axial force and torque (combined tension–torsion) on a thin-walled tube of isotropic polypropylene, the strains in the tube being small. Some results of these experiments have been already presented elsewhere in another form (Buckley & McCrum 1974). In fact, the present theory is used to give a rational basis to the intuitive interpretation which was applied before to these results. Finally we illustrate how stress dependence of the material response functions can be examined systematically by varying independently the two stress invariants involved.

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2. Membrane theory

Let the motion of points of a surface s embedded in Euclidean 3-space be specified by a position vector $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ where

$$\boldsymbol{r} = \boldsymbol{r}(\theta^1, \theta^2, t), \quad \boldsymbol{v} = \dot{\boldsymbol{r}}. \tag{2.1}$$

In (2.1), θ^α $(\alpha = 1, 2)$ are convected coordinates, t is the time and a superposed dot denotes partial differentiation with respect to t holding θ^α fixed. Base vectors $\boldsymbol{a}_\alpha$, $\boldsymbol{a}^\alpha$, unit normal $\boldsymbol{a}_3$, metric tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ and Christoffel symbols are defined by†

$$\left.\begin{aligned} \boldsymbol{a}_\alpha = \partial\boldsymbol{r}/\partial\theta^\alpha, \quad \boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta, \quad \boldsymbol{a}_3 a^{\frac{1}{2}} = \boldsymbol{a}_1\times\boldsymbol{a}_2, \\ a_{\alpha\beta} = \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta, \quad a = \det a_{\alpha\beta}, \quad \Gamma^\alpha_{\beta\gamma} = \boldsymbol{a}^\alpha\cdot\frac{\partial\boldsymbol{a}_\beta}{\partial\theta^\gamma}, \end{aligned}\right\} \tag{2.2}$$

where δ^α_β is the Kronecker delta. Greek indices take the values 1, 2.

Let $\mathscr{P}$, bounded by a closed curve $\partial\mathscr{P}$, be part of the surface s occupied by an arbitrary material region of the surface in the configuration at time t, and let

$$\boldsymbol{v} = \nu_\alpha \boldsymbol{a}^\alpha \tag{2.3}$$

be the outward unit normal to the closed curve $\partial\mathscr{P}$, in the surface. The mass density of the surface in the configuration at time t is $\rho = \rho(\theta^\alpha, t)$. Forces acting on $\mathscr{P}$ are analogous to those acting on a three-dimensional region. They consist of a contact force acting across $\partial\mathscr{P}$ (which is analogous to the stress vector in three dimensions) and which is denoted by $\boldsymbol{n} = \boldsymbol{n}(\theta^\alpha, t; \boldsymbol{v})$ per unit length. Also there is an externally applied force $\boldsymbol{b} = \boldsymbol{b}(\theta^\alpha, t)$ per unit mass of the surface which consists of a body force and the resultant of the forces acting on each side of the surface.

Equations of mass conservation, linear momentum and moment of momentum are‡

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\,\mathrm{d}\sigma = 0, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\boldsymbol{v}\,\mathrm{d}\sigma = \int_{\mathscr{P}} \rho\boldsymbol{b}\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{n}\,\mathrm{d}s, \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \boldsymbol{r}\times\boldsymbol{v}\rho\,\mathrm{d}\sigma = \int_{\mathscr{P}} \boldsymbol{r}\times\boldsymbol{b}\rho\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{r}\times\boldsymbol{n}\,\mathrm{d}s, \tag{2.6}$$

where $\mathrm{d}\sigma$, $\mathrm{d}s$ are elements of surface area and length, respectively. The field equations which may be derived from (2.4) to (2.6) are

$$\rho a^{\frac{1}{2}} = k(\theta^\alpha) = \rho_0 A^{\frac{1}{2}}, \tag{2.7}$$

where ρ_0 and A are the values of ρ and a respectively in some reference configuration of the surface s, and k is independent of t. Also

$$\partial(\boldsymbol{N}^\alpha a^{\frac{1}{2}})/\partial\theta^\alpha + k\boldsymbol{b} = k\dot{\boldsymbol{v}}, \tag{2.8}$$

$$\boldsymbol{a}_\alpha\times\boldsymbol{N}^\alpha = \boldsymbol{0}, \tag{2.9}$$

where

$$\boldsymbol{n} = \boldsymbol{N}^\alpha \nu_\alpha. \tag{2.10}$$

Greek indices which occur twice are summed. If

$$\boldsymbol{N}^\alpha = N^{\alpha i}\boldsymbol{a}_i \tag{2.11}$$

where i is summed over values 1, 2, 3, then (2.9) and (2.8) yield the component equations

$$N^{\alpha 3} = 0, \quad N^{\alpha\beta} = N^{\beta\alpha}, \tag{2.12}$$

$$N^{\alpha\beta}|_\alpha + \rho\boldsymbol{b}\cdot\boldsymbol{a}^\beta = \rho\dot{\boldsymbol{v}}\cdot\boldsymbol{a}^\beta, \quad b_{\alpha\beta}N^{\alpha\beta}a^{\frac{1}{2}} + k\boldsymbol{b}\cdot\boldsymbol{a}^3 = k\dot{\boldsymbol{v}}\cdot\boldsymbol{a}^3, \tag{2.13}$$

† For a summary of surface geometry see Green & Zerna (1968).

‡ The corresponding field equations were obtained by a different procedure by Green, Naghdi & Wainwright (1965). See also Naghdi (1972).

where $\boldsymbol{a}^3 = \boldsymbol{a}_3$, $b_{\alpha\beta}$ is the curvature tensor and a vertical rule denotes surface covariant differentiation. Also (2.11) reduces to

$$\boldsymbol{N}^\alpha = N^{\alpha\beta}\boldsymbol{a}_\beta. \tag{2.14}$$

We assume that the inertia terms in the equations of motion (2.13) may be neglected at the time of measurement of deformation. Also, we assume that the only force applied to the surface s is a pressure difference p, per unit area of s. Then

$$k\boldsymbol{b} = a^{\frac{1}{2}}p\boldsymbol{a}_3 \tag{2.15}$$

and equations (2.13) reduce to

$$N^{\alpha\beta}|_\alpha = 0, \quad b_{\alpha\beta}N^{\alpha\beta} + p = 0. \tag{2.16}$$

In addition to the symmetric two dimensional stress tensor $N^{\alpha\beta}$ we require a symmetric two dimensional strain tensor which we take to be

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \tag{2.17}$$

where $A_{\alpha\beta}$ are the values of $a_{\alpha\beta}$ in a reference configuration. The tensors $N^{\alpha\beta}$, $e_{\alpha\beta}$, $a_{\alpha\beta}$ are all unaltered by rigid-body motions of the whole surface s.

3. Circular cylindrical tube

Consider a circular cylindrical tube of viscoelastic material which is initially unstressed and whose wall thickness is small compared with its length l_0 and radius c_0. We limit attention to experiments in which a resultant force F is applied over the ends of the tube parallel to its axis, a resultant couple G about its axis and a pressure difference p between inside and outside surfaces of the tube, measured per unit area of the deformed tube. All three quantities F, G, p are specified functions of time. Since the wall thickness of the shell is small, and the shell is not under flexure, we regard the shell as a membrane and make use of the theory in § 2. We assume that F, G and p are applied in such a way that homogeneous strains are induced in the tube so that at time t the radius and length become $c(t)$ and $l(t)$ respectively, where

$$c(t) = c_0\mu(t), \quad l(t) = l_0\lambda(t). \tag{3.1}$$

In addition there is a twist $\psi(t)$ per unit length of the cylinder at time t.

The tube in a reference configuration at time t_0 and in the configuration at time $t > t_0$, may be described by position vectors $\boldsymbol{R}$, $\boldsymbol{r}$ respectively where

$$\left.\begin{aligned} \boldsymbol{R} &= z\boldsymbol{e}_3 + c_0(\boldsymbol{e}_1\cos\theta + \boldsymbol{e}_2\sin\theta), \\ \boldsymbol{r} &= \lambda z\boldsymbol{e}_3 + \mu c_0[\boldsymbol{e}_1\cos(\theta+\psi\lambda z) + \boldsymbol{e}_2\sin(\theta+\psi\lambda z)], \end{aligned}\right\} \tag{3.2}$$

where $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ is a system of constant orthonormal vectors and

$$0 \leqslant z \leqslant l_0, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{3.3}$$

We choose the coordinates θ^α so that

$$\theta^1 = \theta c_0, \quad \theta^2 = z. \tag{3.4}$$

Then from (2.2) and (3.2) we have

$$\left.\begin{aligned} \boldsymbol{a}_1 &= \mu[-\boldsymbol{e}_1\sin(\theta+\psi\lambda z) + \boldsymbol{e}_2\cos(\theta+\psi\lambda z)], \\ \boldsymbol{a}_2 &= \mu\lambda\psi c_0[-\boldsymbol{e}_1\sin(\theta+\psi\lambda z) + \boldsymbol{e}_2\cos(\theta+\psi\lambda z)] + \lambda\boldsymbol{e}_3, \\ \boldsymbol{a}_3 &= \boldsymbol{e}_1\cos(\theta+\psi\lambda z) + \boldsymbol{e}_2\sin(\theta+\psi\lambda z). \end{aligned}\right\} \tag{3.5}$$

Hence
$$\left.\begin{aligned} a_{11} = \mu^2, \quad a_{12} = \mu^2\lambda\psi c_0, \quad a_{22} = \lambda^2(1+\mu^2\psi^2c_0^2), \quad \Gamma^\alpha_{\beta\gamma} = 0, \\ a = \lambda^2\mu^2, \quad a^{11} = \mu^{-2}+\psi^2c_0^2, \quad a^{12} = -\psi c_0/\lambda, \quad a^{22} = \lambda^{-2}. \end{aligned}\right\} \tag{3.6}$$

Moreover, since
$$b_{\alpha\beta} = \boldsymbol{a}_3\cdot\partial\boldsymbol{a}_\alpha/\partial\theta^\beta, \quad b^\alpha_\beta = a^{\alpha\lambda}b_{\lambda\beta}, \tag{3.7}$$
it follows that
$$\left.\begin{aligned} b_{11} = -\mu/c_0, \quad b_{12} = -\mu\lambda\psi, \quad b_{22} = -\mu\lambda^2\psi^2c_0, \\ b^1_1 = -\frac{1}{\mu c_0}, \quad b^1_2 = -\frac{\lambda\psi}{\mu}, \quad b^2_1 = b^2_2 = 0. \end{aligned}\right\} \tag{3.8}$$

The values of $a_{\alpha\beta}$, $a^{\alpha\beta}$, a in the reference configuration are denoted by $A_{\alpha\beta}$, $A^{\alpha\beta}$, A respectively and are given by
$$A_{\alpha\beta} = A^{\alpha\beta} = \delta^\alpha_\beta, \quad A = 1. \tag{3.9}$$

From (2.17), (3.6) and (3.9) we obtain the components of strain
$$e_{11} = \tfrac{1}{2}(\mu^2-1), \quad e_{22} = \tfrac{1}{2}[\lambda^2(1+\mu^2\psi^2c_0^2)-1], \quad e_{12} = \tfrac{1}{2}\mu^2\lambda\psi c_0. \tag{3.10}$$

The unit normal to a curve z = constant in the configuration at time t is given by
$$\boldsymbol{\nu} = \boldsymbol{a}^2/(a^{22})^{\frac{1}{2}}, \quad \nu_1 = 0, \quad \nu_2 = (a^{22})^{-\frac{1}{2}} = \lambda. \tag{3.11}$$

From (2.11) and (3.11) it follows that the force vector across this curve is
$$\begin{aligned} \boldsymbol{n} &= \lambda N^{2\beta}\boldsymbol{a}_\beta \\ &= \lambda^2N^{22}\boldsymbol{e}_3+\lambda\mu(N^{21}+\lambda\psi c_0N^{22})\,[-\boldsymbol{e}_1\sin(\theta+\psi\lambda z)+\boldsymbol{e}_2\cos(\theta+\psi\lambda z)] \end{aligned} \tag{3.12}$$
per unit length of the curve in the configuration at time t. Since the line element is given by
$$\mathrm{d}s = (a_{11})^{\frac{1}{2}}\,\mathrm{d}\theta^1 = \mu c_0\,\mathrm{d}\theta, \tag{3.13}$$
we obtain from (3.12) the formulae
$$F = 2\pi\lambda^2\mu c_0N^{22}, \quad G = 2\pi\lambda\mu^3c_0^2(N^{21}+\lambda\psi c_0N^{22}). \tag{3.14}$$

We consider a viscoelastic material in which the strain at time t depends on the whole history of the stress $N^{\alpha\beta}$. Since the strain in (3.10) and the metric tensor (3.6) are functions of t and are independent of θ^α, we consider only stress histories which are independent of θ^α. It follows that the equations of motion (2.16) are then satisfied provided
$$p = \mu c_0^{-1}[N^{11}+2\lambda\psi c_0N^{12}+\lambda^2\psi^2c_0^2N^{22}]. \tag{3.15}$$

If $N^{\alpha\beta}$ has a specified history as a function of time then it is clear that the deformation discussed here can only be maintained if F, G, p have the values given in (3.14) and (3.15) which depend also on λ, μ, ψc_0. Under experimental conditions we wish to be able to specify the history of F, G and p. Also, in order to reduce the theory to a more usable form it is desirable to be able to choose stress histories of a proportional loading type, or possibly a combination of such histories, but this is, in general, incompatible with arbitrary choice of F, G and p in the present theory.

In many experiments with viscoelastic materials such as polypropylene it is found that the response of the material is nonlinear even when the strains in the tube are small. In such a situation we may introduce an approximation for λ, μ, ψc_0 in (3.14), (3.15) and the strain–stress relations, to yield a theory which meets the requirements mentioned above. If
$$\lambda = 1+\epsilon_l, \quad \mu = 1+\epsilon_c, \quad \psi c_0 = \gamma/\lambda, \tag{3.16}$$

and if the numbers ϵ_l, ϵ_c, γ which represent the deformation of the tube are small, then we have, approximately†

$$F = 2\pi c_0 N^{22}, \quad G = 2\pi c_0^2 N^{12}, \quad p = c_0^{-1} N^{11} \tag{3.17}$$

and

$$e_{11} = \epsilon_c, \quad e_{22} = \epsilon_l, \quad e_{12} = \tfrac{1}{2}\gamma. \tag{3.18}$$

If we now specify values of F, G, p as functions of time we control the values of $N^{\alpha\beta}$ independently of the strain. With the same approximation we may form covariant components $N_{\alpha\beta}$ and mixed components N^{α}_{β} of stress from $N^{\alpha\beta}$ with the help of the metric tensor $A_{\alpha\beta}$ instead of $a_{\alpha\beta}$, so that

$$N^{\alpha}_{\beta} = A_{\beta\lambda} N^{\alpha\lambda}, \quad N_{\alpha\beta} = A_{\alpha\lambda} N^{\lambda}_{\beta}. \tag{3.19}$$

4. Stress histories

It is convenient to work with invariant forms of the stress and strain tensors which are defined in terms of $e_{\alpha\beta}$, $N^{\alpha\beta}$ by the equations

$$\boldsymbol{E} = e_{\alpha\beta} \boldsymbol{A}^{\alpha} \otimes \boldsymbol{A}^{\beta}, \quad \boldsymbol{N} = N^{\alpha\beta} \boldsymbol{A}_{\alpha} \otimes \boldsymbol{A}_{\beta}. \tag{4.1}$$

The symmetric tensors $\boldsymbol{E}$, $\boldsymbol{N}$ are unaltered by superposed rigid body motions of the whole surface. In the light of the small strain approximation introduced at the end of § 3 we assume that the surface consists of a simple material which is such that the strain $\boldsymbol{E}$ depends on the whole history of the stress $\boldsymbol{N}$. Thus

$$\boldsymbol{E} = \mathop{\hat{\boldsymbol{E}}}_{\tau=-\infty}^{t} [\boldsymbol{N}(\tau), t-\tau] \tag{4.2}$$

where $\hat{\boldsymbol{E}}$ is a functional of the stress history.‡ Moreover, if the surface is isotropic, with a centre of symmetry, in its reference configuration, the functional in (4.2) is isotropic. Also $\boldsymbol{E}$ given by (4.2) must form a compatible system of strain.

For the problem of the cylindrical tube discussed in § 3 we wish to be able to specify F, G and p as functions of the time so that, from (3.17), $N^{\alpha\beta}$ must be functions of time only when we make the particular choice of coordinates (3.4). Moreover the base tensors $\boldsymbol{A}_{\alpha} \otimes \boldsymbol{A}_{\beta}$ are functions of θ^{λ}, independent of time. It follows from $(4.1)_2$ that for the problems under discussion in which F, G and p are specified as functions of time the tensor $\boldsymbol{N}(\tau)$ can always be expressed in the form

$$\boldsymbol{N}(\tau) = \boldsymbol{P}(\theta^{\alpha}) f(\tau) + \boldsymbol{Q}(\theta^{\alpha})\, g(\tau) + \boldsymbol{R}(\theta^{\alpha})\, q(\tau), \tag{4.3}$$

where $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are symmetric tensor functions of θ^{α} independent of time τ, and f, g, q are functions of τ. If any one of F, G and p is maintained at zero it is always possible to take $\boldsymbol{R} = \boldsymbol{0}$ in (4.3), while if any two are maintained at zero it is always possible to take $\boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{0}$. In general we adopt the form (4.3) for $\boldsymbol{N}(\tau)$ and substitute into (4.2), which then reduces to

$$\boldsymbol{E} = \overline{\boldsymbol{E}}(\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}), \tag{4.4}$$

where $\overline{\boldsymbol{E}}$ is a symmetric isotropic function of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and also a functional of $f(\tau)$, $g(\tau)$, $q(\tau)$. Since we are dealing with two dimensional symmetric tensors it follows (Rivlin 1955; Pipkin & Wineman 1963, 1964) that

$$\boldsymbol{E} = \phi_0 \boldsymbol{I} + \phi_1 \boldsymbol{P} + \phi_2 \boldsymbol{Q} + \phi_3 \boldsymbol{R} \tag{4.5}$$

† The numbers ϵ_l, ϵ_c and γ may be recognized as the 'engineering' tensile strains in longitudinal and circumferential directions and shear strain in the wall of the tube, respectively. The two dimensional stresses N^{22}, N^{11}, N^{12} represent the corresponding three dimensional 'engineering' stresses, simply multiplied by the initial wall thickness of the tube.

‡ This assumption implies that we are neglecting the effect on $\boldsymbol{E}$ of the mean pressure through the tube. The response functional is invariant under a linear time change $t^* = t+k$, $\tau^* = \tau+k$, $\boldsymbol{N}(\tau^*) = \boldsymbol{N}(\tau)$, where k is a constant.

where $\boldsymbol{I}$ is the unit tensor and ϕ_0, ϕ_1, ϕ_2, ϕ_3 are scalar functions of the invariants

$$\left.\begin{array}{c}\operatorname{tr}\boldsymbol{P},\quad \operatorname{tr}\boldsymbol{Q},\quad \operatorname{tr}\boldsymbol{R},\quad \operatorname{tr}\boldsymbol{P}^2,\quad \operatorname{tr}\boldsymbol{Q}^2,\quad \operatorname{tr}\boldsymbol{R}^2,\\ \operatorname{tr}\boldsymbol{PQ},\quad \operatorname{tr}\boldsymbol{QR},\quad \operatorname{tr}\boldsymbol{PR},\end{array}\right\} \tag{4.6}$$

and functionals of $f(\tau)$, $g(\tau)$, $q(\tau)$.

Although the above results are sufficient for any problem considered in § 3 in which F, G and p are specified in any way we choose, we limit our attention in the rest of the paper to situations of proportional loading, i.e. those in which we can take $\boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{0}$. Then (4.5) becomes

$$\boldsymbol{E} = \phi_0\boldsymbol{I}+\phi_1\boldsymbol{P} \tag{4.7}$$

where ϕ_0, ϕ_1 are functions of $\quad \operatorname{tr}\boldsymbol{P},\quad \operatorname{tr}\boldsymbol{P}^2,$ (4.8)

and functionals of $f(\tau)$. For the small strain approximation in which we may use (3.17) and (3.18) we write

$$F = 2\pi c_0 T(t),\quad G = 2\pi c_0^2 S(t),\quad c_0 p = P^{11}f(t),\quad S(t) = P^{12}f(t),\quad T(t) = P^{22}f(t), \tag{4.9}$$

where $P^{\alpha\beta}$ are constants. For this to be possible the ratios

$$c_0 p(t){:}\ S(t){:}\ T(t) = P^{11}{:}\,P^{12}{:}\,P^{22} \tag{4.10}$$

must be maintained constant throughout the experiment although these ratios are at our choice by assigning suitable constant values to the contravariant components of the tensor $\boldsymbol{P}$. From (3.9), (3.18), (4.7) to (4.9) we see that

$$\epsilon_c = \phi_0+\phi_1 P^{11},\quad \epsilon_l = \phi_0+\phi_1 P^{22},\quad \tfrac{1}{2}\gamma = \phi_1 P^{12}, \tag{4.11}$$

where ϕ_0, ϕ_1 are scalar functionals of $f(\tau)$ and functions of the invariants

$$I_1 = \operatorname{tr}\boldsymbol{P} = P^{11}+P^{22},\quad I_2 = \operatorname{tr}\boldsymbol{P}^2 = (P^{11})^2+(P^{22})^2+2(P^{12})^2. \tag{4.12}$$

It is interesting to note from (4.11) that the strains are related at all times t through the formula

$$\frac{2[\epsilon_l(t)-\epsilon_c(t)]}{\gamma(t)} = \frac{P^{22}-P^{11}}{P^{12}} = \frac{T(t)-c_0 p(t)}{S(t)}, \tag{4.13}$$

which does not depend explicitly on the material response functions.

The values p, S, T may be maintained separately constant if we choose

$$f(t) = h(t-t_1)\quad (t_1 \leqslant t) \tag{4.14}$$

where $h(t)$ is the Heaviside unit function.

In all the experiments considered later the pressure difference p on the surfaces of the tube is zero so that (4.11) to (4.13) reduce to

$$\left.\begin{array}{c}\epsilon_c = \phi_0,\quad \epsilon_l = \phi_0+\phi_1 P^{22},\quad \tfrac{1}{2}\gamma = \phi_1 P^{12},\\ \dfrac{2[\epsilon_l(t)-\epsilon_c(t)]}{\gamma(t)} = \dfrac{P^{22}}{P^{12}} = \dfrac{T(t)}{S(t)},\end{array}\right\} \tag{4.15}$$

where ϕ_0, ϕ_1 are functionals of $f(\tau)$ and functions of

$$I_1 = P^{22},\quad I_2 = (P^{22})^2+2(P^{12})^2. \tag{4.16}$$

The experimental results described in later sections are related to theory through (4.15) and (4.16).

5. Experimental details

5.1. *Apparatus*

The combined tension–torsion creep apparatus used in this work has been described in detail by Hutchinson (1974). Its principle of operation is as follows.

The specimen has the form of a thin-walled circular cylindrical tube with integral flanges at each end. It is mounted vertically with the bottom flange rigidly fixed. An axial force F may be applied by means of weights suspended from a lever loading arm. An axial couple G may be applied by passing current through a torque coil positioned between the poles of a permanent magnet. Elongation of the specimen is measured by an l.v.d.t. transducer which monitors vertical movement of the top flange. Rotation of the top flange is measured by following the angular deflexion of a light beam, reflected from a mirror which moves with the flange. Movement of the light beam is followed along a linear scale normal to the beam, a correction being made for the small tangent error involved (less than 0.75 %).

In the work described here we shall assume that the deformation of the specimen between its flanges is homogeneous and can therefore be simply deduced from the axial displacement and rotation of the top flange. To reduce the effect of stress concentration, which would invalidate this assumption, the specimen is flared with 4 mm radius shoulders at the junctions of the uniform tube section with each end flange. It was found during preliminary tests with isotropic polypropylene specimens of different lengths l_0 (distance between flanges) that this geometry causes end-effects which appear only as an apparent reduction of l_0 by 2.2 mm, for strains of up to *ca.* 0.01. The apparent stress–strain behaviour of the specimen is otherwise independent of l_0, justifying use of the above assumption when studying nonlinear viscoelastic behaviour.

In spite of careful alignment of the apparatus it was found that a small, but significant, degree of cross-coupling occurred between tensile and torsional operating modes. This was calibrated using creep experiments in pure tension and pure torsion, and allowances made for the effect in combined tension–torsion experiments described below. Thus the force F was found to induce a couple G_i of order $-0.016c_0F$, while a shear strain γ was found to give an apparent tensile strain ϵ_{li} of order -0.012γ.

With the specimen mounted in the apparatus there are seven possible contributors to random scatter between results. These are non-reproducibility of transducer calibration (a), ϵ_{li}(b) and G_i(c), and measuring errors in the transducer output (d), light beam scale position (e), mirror-to-scale throw of the light beam (f) and torque coil current (g). The possible magnitudes of these errors are indicated where appropriate in the results presented below.

5.2. *Specimen*

All experiments reported here were performed using a single specimen, machined from an isotropic 60 mm diameter cylindrical bar of polypropylene (Propathene PXC 8830†). Mechanical isotropy of the bar was confirmed by conducting tensile creep tests on rectangular specimens (of dimensions 1 mm × 3 mm × 33 mm) machined with their long axes parallel or perpendicular to the axis of the bar (two specimens of each). The difference between mean values of the tensile creep compliance D(10 s) for the two directions amounted to only 2 %, which cannot be considered significant.

† Supplied by courtesy of I.C.I. Limited.

Dimensions of the tube specimen were measured at 20 °C and corrected to 65.5 °C using a coefficient of linear thermal expansion of 1.35×10^{-4} (°C)$^{-1}$, to give length (l_0) 65.4 mm, radius (c_0) 4.28 mm and wall thickness (w_0) 0.427 mm.

Following machining, it was annealed within the creep apparatus at 100 °C for 6 h to relieve any stresses induced by machining. No detectable distortion or changes in dimensions occurred. When remounted in the apparatus it was surrounded by a closed copper jacket immersed in a temperature-controlled water bath. The temperature was raised to 65.5 °C and maintained for three weeks before beginning the experimental programmes described below. The experiments took place over a period of 6 months, during which time the temperature (as measured by a thermocouple close to the specimen) varied from 65.5 °C by less than 0.1 °C, except for occasional drops of *ca.* 5 °C (of *ca.* $\frac{1}{2}$ h duration) when the bath was unavoidably replenished with cold water. This procedure was necessary to ensure that the specimen was thermally well equilibrated and that any internal thermal stresses (Hutchinson & McCrum 1972) had sufficient time to relax. The results of different experiments could therefore be meaningfully compared, even if separated by periods of recovery of days or weeks. In addition, care was taken that strains ϵ_l, ϵ_c, γ never exceeded 0.012, to avoid causing permanent damage to the specimen.

As a result of these precautions, over the 6 month period, no significant change (greater than *ca.* 1 %) occurred in linear viscoelastic response, while at the highest stresses employed ageing of the material was only just detectable, as a fractional decrease of 2 % in the strain response to a given applied stress. Over periods of up to one month the results reported below were found to be entirely reproducible to within the precision of measurement.

6. Results and discussion

6.1. *Isochronous stress–strain data for creep and recovery*

Since the aim of the present work was to study departures from linear viscoelasticity in the behaviour of the tube, the first step was to measure its linear viscoelastic response, pertaining to small stresses. This was achieved by using creep experiments in pure tension and pure torsion. In each case the loading history consisted of m regularly increasing rectangular pulses of F (or G) of duration $t_c = 120$ s, separated by recovery periods of duration $t_r = 1200$ s. Thus the pure tension experiment could be described by

$$T(\tau) = \sum_{n=1}^{m} nT_0 j_n(\tau), \quad S(\tau) = 0, \quad p(\tau) = 0, \tag{6.1}$$

where T_0 is constant, and

$$j_n(\tau) = h[\tau-(n-1)(t_c+t_r)] - h[\tau-(n-1)(t_c+t_r)-t_c]. \tag{6.2}$$

The pure torsion experiment is given by

$$S(\tau) = \sum_{n=1}^{m'} nS_0 j_n(\tau), \quad T(\tau) = 0, \quad p(\tau) = 0, \tag{6.3}$$

where S_0 is constant. The origin of τ has been chosen in each case to coincide with the onset of the first pulse. In practice, the specimen was found to have a fading memory, such that with these values of t_c and t_r the response to a stress pulse of given magnitude was independent of the number and magnitude of previous pulses.† In this case the individual pulses can be considered separately and so described merely by

$$T(\tau) = T[h(\tau)-h(\tau-t_c)], \quad S(\tau) = 0, \quad p(\tau) = 0, \tag{6.4}$$

for pure tension, where T is constant, and

$$S(\tau) = S[h(\tau)-h(\tau-t_c)], \quad T(\tau) = 0, \quad p(\tau) = 0, \tag{6.5}$$

for pure torsion where S is constant. Since (6.4) and (6.5) each correspond to special cases of (4.9) with $p = 0$, the creep response at time t ($t < t_c$) after loading with a pulse of magnitude T (or S) is given from (4.15) and (4.16) by

$$\left.\begin{aligned} \epsilon_l^c(T,t) &= \phi_0^c(I_1,I_2,t) + T\phi_1^c(I_1,I_2,t), \\ \epsilon_c^c(T,t) &= \phi_0^c(I_1,I_2,t), \quad (t < t_c), \end{aligned}\right\} \tag{6.6}$$

where $I_1 = T$, $I_2 = T^2$, for tension, and

$$\gamma^c(S,t) = 2S\phi_1^c(I_1\ I_2,\ t), \quad (t < t_c), \tag{6.7}$$

where $I_1 = 0$, $I_2 = 2S^2$, for torsion, a superscript c referring to a creep test. For the special case of linear viscoelastic response, (6.6) and (6.7) yield

$$\left.\begin{aligned} &\lim_{T\to 0} \epsilon_c^c(T,t)/T = \phi_0^{c*}(t), \\ &\lim_{T\to 0} \epsilon_l^c(T,\ t)/T = \phi_0^{c*}(t) + \phi_1^{c*}(t), \\ &\lim_{S\to 0} \gamma^c(S,\ t)/S = 2\phi_1^{c*}(t). \end{aligned}\right\} \tag{6.8}$$

In (6.8),

$$\left.\begin{aligned} \phi_0^{c*}(t) &= \lim_{I_1\to 0} \phi_0^c(I_1,\ 0,\ t)/I_1, \\ \phi_1^{c*}(t) &= \phi_1^c(0,\ 0,\ t), \end{aligned}\right\} \tag{6.9}$$

are independent of stress and characterize the linear viscoelastic creep response.

The results of these experiments are shown in figures 1 and 2 as isochronous plots for $t = 10$ s of ϵ_l^c/T against T and $\frac{1}{2}\gamma^c/S$ against S for pure tension and pure torsion respectively. Error bars correspond to possible random error from sources (*a*), (*d*)–(*g*) mentioned above. Extrapolating the curves of figures 1 and 2 as shown gives, from (6.8)

$$\left.\begin{aligned} \phi_0^{c*}(10\,\mathrm{s}) &= -1.207\,\mathrm{m/MN}, \\ \phi_1^{c*}(10\,\mathrm{s}) &= 4.173\,\mathrm{m/MN}. \end{aligned}\right\} \tag{6.10}$$

Creep response under combined tension–torsion was determined by using similar loading programmes, each of the form

$$\left.\begin{aligned} T(\tau) &= \sum_{n=1}^{m} nT_0 j_n(\tau), \\ S(\tau) &= \sum_{n=1}^{m} nS_0 j_n(\tau), \end{aligned}\right\} \tag{6.11}$$

each programme being characterized by the value of the ratio $T_0 : S_0$ where T_0, S_0 are constants. Again t_c and t_r were chosen to be 120 s and 1200 s respectively, and because of the fading memory

† This was checked by varying the sequence of loading from the regular pattern given by (6.1) and (6.3) – with unchanged t_c and t_r. No difference could be detected in the creep response to a given stress.

of the specimen the response to (6.11) was indistinguishable from the response to isolated stress pulses given by $p(\tau) = 0$, and

$$\left.\begin{aligned} T(\tau) &= T[h(\tau)-h(\tau-t_c)],\\ S(\tau) &= S[h(\tau)-h(\tau-t_c)],\end{aligned}\right\} \tag{6.12}$$

where T, S are constants.

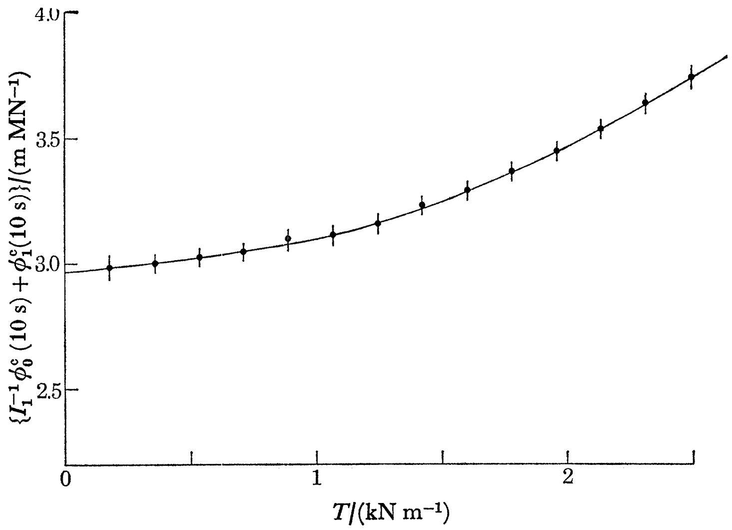

FIGURE 1. Creep under pure tension. Isochronal plot of $\epsilon_l^c(t)/T$ against T for $t = 10$ s.

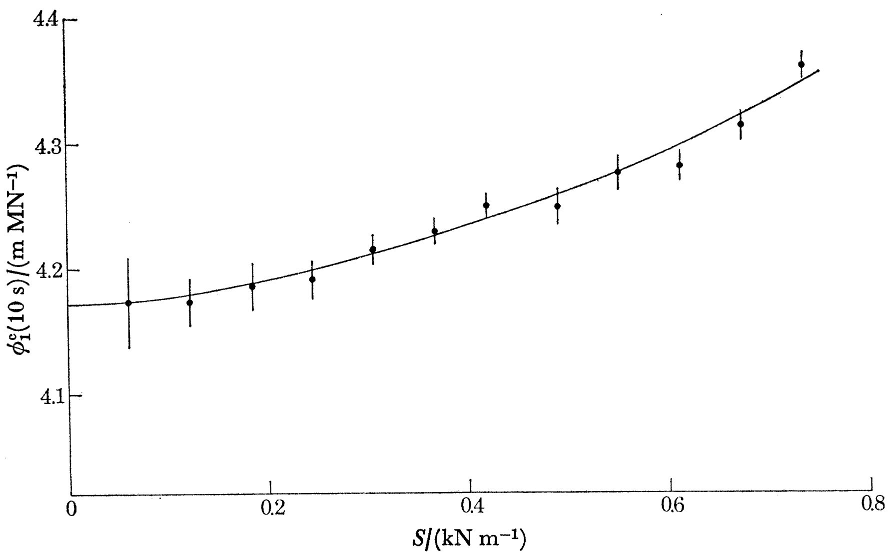

FIGURE 2. Creep under pure torsion. Isochronal plot of $\frac{1}{2}\gamma^c(t)/S$ against S for $t = 10$ s.

The forms (6.12) are a special case of (4.9) in which

$$f(\tau) = h(\tau)-h(\tau-t_c), \quad P^{12} = S, \quad P^{22} = T. \tag{6.13}$$

The corresponding strains computed from (4.15) are

$$\left.\begin{aligned} \epsilon_l^c(T,S,t) &= \phi_0^c(I_1,I_2,t)+T\phi_1^c(I_1,I_2,t),\\ \epsilon_c^c(T,S,t) &= \phi_0^c(I_1,I_2,t),\\ \gamma^c(T,S,t) &= 2S\phi_1^c(I_1,I_2,t),\end{aligned}\right\} \tag{6.14}$$

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for $t < t_c$ and

$$\left.\begin{aligned} \epsilon_l^{\mathrm{r}}(T, S, t-t_c, t_c) &= \phi_0^{\mathrm{r}}(I_1, I_2, t-t_c, t_c) + T\phi_1^{\mathrm{r}}(I_1, I_2, t-t_c, t_c), \\ \epsilon_c^{\mathrm{r}}(T, S, t-t_c, t_c) &= \phi_0^{\mathrm{r}}(I_1, I_2, t-t_c, t_c), \\ \gamma^{\mathrm{r}}(T, S, t-t_c, t_c) &= 2S\phi_1^{\mathrm{r}}(I_1, I_2, t-t_c, t_c), \end{aligned}\right\} \tag{6.15}$$

for $t_c < t$, where from (4.16) and (6.13),

$$I_1 = T, \quad I_2 = T^2 + 2S^2. \tag{6.16}$$

A superscript r refers to recovery after stress is removed. The formula $(4.15)_4$ follows and was used to calculate the strain ϵ_c from the measured strains ϵ_l and γ. Results during creep are shown in figure 3 as isochronous plots of ϵ_l^{c}, ϵ_c^{c}, γ^{c} against T and S, for $t = 10$ s and different values of the ratios T:S. Included in figure 3 for comparison are straight lines corresponding to linear visco-elastic behaviour

$$\left.\begin{aligned} \epsilon_c^{\mathrm{c}}(10\,\mathrm{s}) &= T\phi_0^{\mathrm{c}*}(10\,\mathrm{s}), \\ \epsilon_l^{\mathrm{c}}(10\,\mathrm{s}) &= T[\phi_0^{\mathrm{c}*}(10\,\mathrm{s}) + \phi_1^{\mathrm{c}*}(10\,\mathrm{s})], \\ \gamma^{\mathrm{c}}(10\,\mathrm{s}) &= 2S\phi_1^{\mathrm{c}*}(10\,\mathrm{s}), \end{aligned}\right\} \tag{6.17}$$

using $\phi_0^{\mathrm{c}*}$, $\phi_1^{\mathrm{c}*}$ from (6.10). The reader will note from figure 3 that all three strain components are below 0.01 under the conditions of these experiments, and so the approximations (3.17) and (3.18) and use of the equations which rely upon them were justified in the present case. Nevertheless, non-linearity in behaviour is clearly discernible, even at strains as low as 0.001. Another interesting feature, the synergistic effect of tension on shear response and vice versa, is also apparent in figure 3 (for example, compare $\gamma^{\mathrm{c}}(t)$ for T:S = 1:2 and 6:1 in figure 3).

We conclude this section by considering recovery of the specimen following creep under combined tension–torsion. At a time $t-t_c$ after unloading let the strain component $e_{\alpha\beta}$ be $e_{\alpha\beta}^{\mathrm{r}}(t-t_c, t_c)$, while if the load had not been removed the same strain component at the same time would have been $e_{\alpha\beta}^{\mathrm{c}}(t)$. We follow the convention of previous authors (Ward & Onat 1963) and define a recovery strain $\Delta e_{\alpha\beta}$

$$\Delta e_{\alpha\beta}(t-t_c, t_c) = e_{\alpha\beta}^{\mathrm{c}}(t) - e_{\alpha\beta}^{\mathrm{r}}(t-t_c, t_c) \tag{6.18}$$

illustrated in figure 4. The significance of $\Delta e_{\alpha\beta}$ is that for linear viscoelastic behaviour

$$\Delta e_{\alpha\beta}(t-t_c, t_c) = e_{\alpha\beta}^{\mathrm{c}}(t-t_c) \tag{6.19}$$

for all t_c. In practice, it is commonly found with nonlinear viscoelastic synthetic polymers that the difference

$$\Delta e_{\alpha\beta}(t-t_c, t_c)/e_{\alpha\beta}^{\mathrm{c}}(t-t_c) - 1$$

is positive for small $t-t_c$, changing to negative at large $t-t_c$ (Turner 1966; Ward & Onat 1963). The experimental loading programmes were modified slightly from (6.11) to allow determination of both $e_{\alpha\beta}^{\mathrm{c}}$ and $e_{\alpha\beta}^{\mathrm{r}}$ in (6.18). Every even-numbered pulse was made of equal magnitude to the previous pulse but of duration $2t_c$ and was followed by a recovery period of $2t_r$. This loading history may be expressed compactly by

$$T(\tau) = \sum_{n=1}^{m} nT_0 k_n(\tau), \quad S(\tau) = \sum_{n=1}^{m} nS_0 k_n(\tau), \tag{6.20}$$

where

$$\begin{aligned} k_n(\tau) = {} & h[\tau - 3(n-1)(t_c+t_r)] - h[\tau - 3(n-1)(t_c+t_r) - t_c] \\ & + h[\tau - (3n-2)(t_c+t_r)] - h[\tau - (3n-2)(t_c+t_r) - 2t_c]. \end{aligned} \tag{6.21}$$

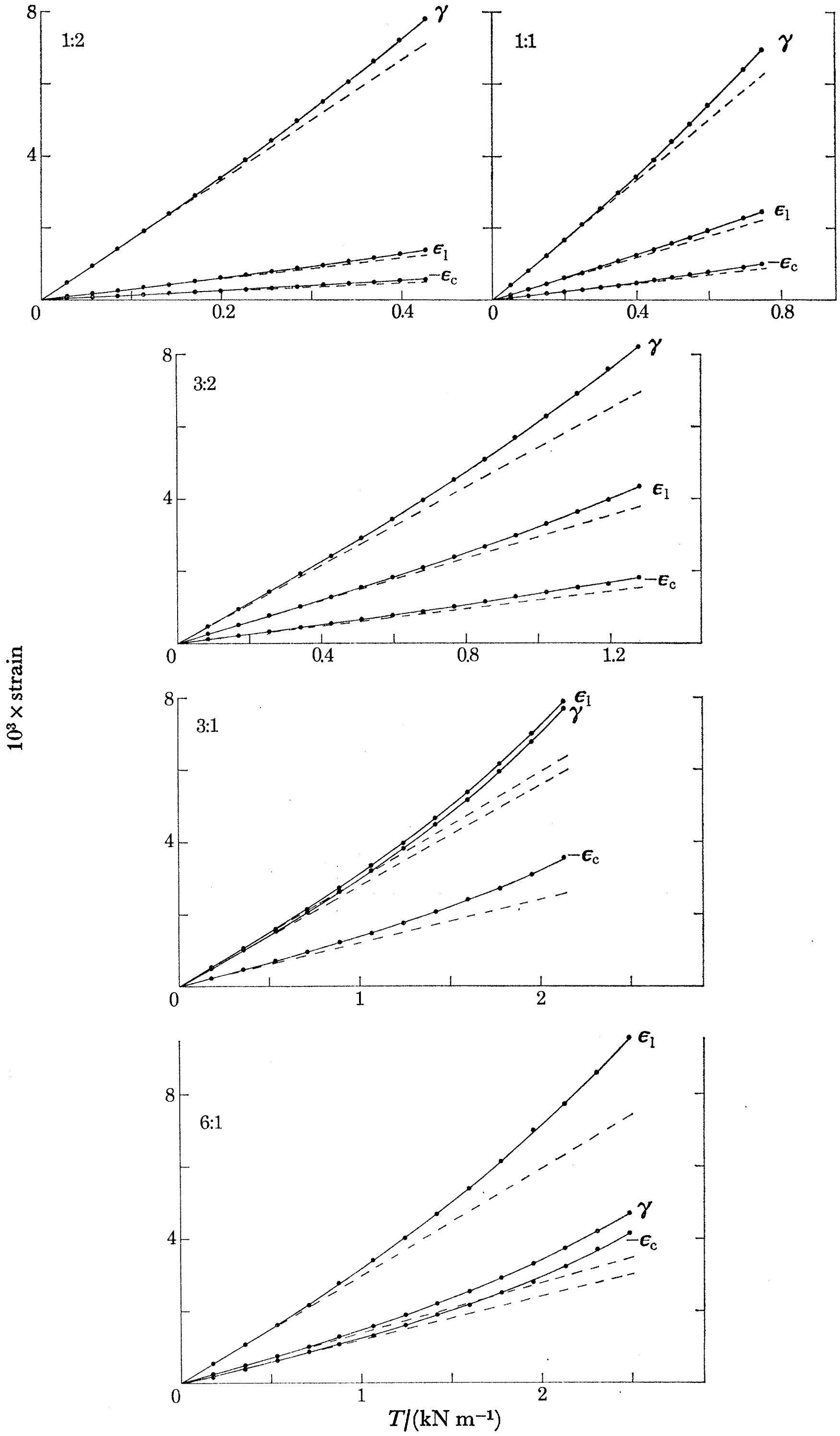

FIGURE 3. Creep under combined tension–torsion. Isochronal plots of creep strain components against stress, for t = 10 s and various values of ratio $T:S$. Circumferential creep strain ϵ_c^c was calculated from ϵ_l^c and γ^c using (4.15). Dashed lines show linear response.

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Parameters t_c, t_r were again chosen to be 120 s and 1200 s respectively, and again the response to (6.20) could be assumed identical to that for isolated pulses of type (6.12) (odd numbered pulses) or

$$T(\tau) = T[h(\tau) - h(\tau - 2t_c)], \quad S(\tau) = S[h(\tau) - h(\tau - 2t_c)] \tag{6.22}$$

(even numbered pulses).

Strains ϵ_l^r, γ^r and hence, from (4.15), ϵ_c^r were measured during recovery from odd pulses, while ϵ_l^c, γ^c and hence ϵ_c^c were measured for the same values of T and S during creep under even pulses. In this way $\Delta\epsilon_l$, $\Delta\gamma$, $\Delta\epsilon_c$ were calculated from (6.18). Isochronous results are plotted against T and S in figure 5 for $t - t_c = 10$ s, $t_c = 120$ s and different values of the ratio $T:S$. We again include, for comparison, the straight lines corresponding to linear viscoelastic behaviour given from (6.17) and (6.19) by

$$\left.\begin{aligned} \Delta\epsilon_c(10\,\text{s}, 120\,\text{s}) &= T\phi_0^{c*}(10\,\text{s}),\\ \Delta\epsilon_l(10\,\text{s}, 120\,\text{s}) &= T[\phi_0^{c*}(10\,\text{s}) + \phi_1^{c*}(10\,\text{s})],\\ \Delta\gamma(10\,\text{s}, 120\,\text{s}) &= 2S\phi_1^{c*}(10\,\text{s}). \end{aligned}\right\} \tag{6.23}$$

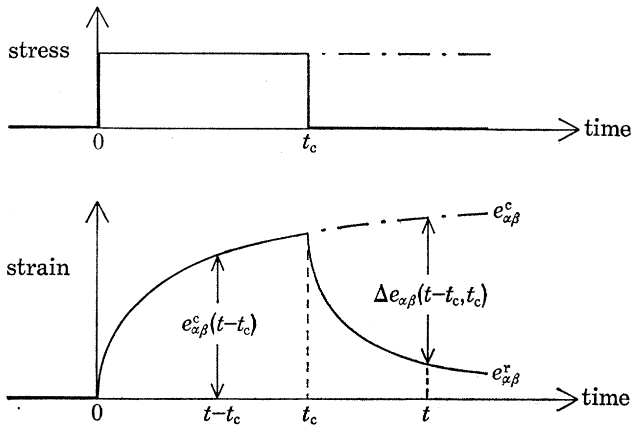

FIGURE 4. Creep and recovery loading programme, showing definition of recovery strain $\Delta e_{\alpha\beta}$.

In general features the results resemble those for creep strains given in figure 3. Comparing figures 3 and 5, however, viscoelastic non-linearity may now be seen to appear as an increase in the magnitudes of $\Delta\epsilon_c(10\,\text{s}, 120\,\text{s}) - \epsilon_c^c(10\,\text{s})$, $\Delta\epsilon_l(10\,\text{s}, 120\,\text{s}) - \epsilon_l^c(10\,\text{s})$, $\Delta\gamma(10\,\text{s}, 120\,\text{s}) - \gamma^c(10\,\text{s})$ with increasing stress.

6.2. *Relation between ϕ_0 and ϕ_1*

It was shown in § 4 that response of the tube to a proportional loading history is completely specified by the two functions ϕ_0, ϕ_1. We now enquire whether these functions are in fact independent for our particular tube of polypropylene.

Consider the creep experiment given by

$$\boldsymbol{N}(\tau) = \boldsymbol{P}[h(\tau) - h(\tau - t_c)]. \tag{6.24}$$

Since (6.24) is a special case of (4.3) with $\boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{0}$, the strain response at time t $(t < t_c)$ is given from (4.7) and (4.8) by

$$\boldsymbol{E}^c(t) = \phi_0^c \boldsymbol{I} + \phi_1^c \boldsymbol{P}, \tag{6.25}$$

where

$$\left.\begin{aligned} \phi_0^c = \phi_0^c(I_1, I_2, t), \quad \phi_1^c = \phi_1^c(I_1, I_2, t),\\ I_1 = \operatorname{tr} \boldsymbol{P}, \quad I_2 = \operatorname{tr} \boldsymbol{P}^2. \end{aligned}\right\} \tag{6.26}$$

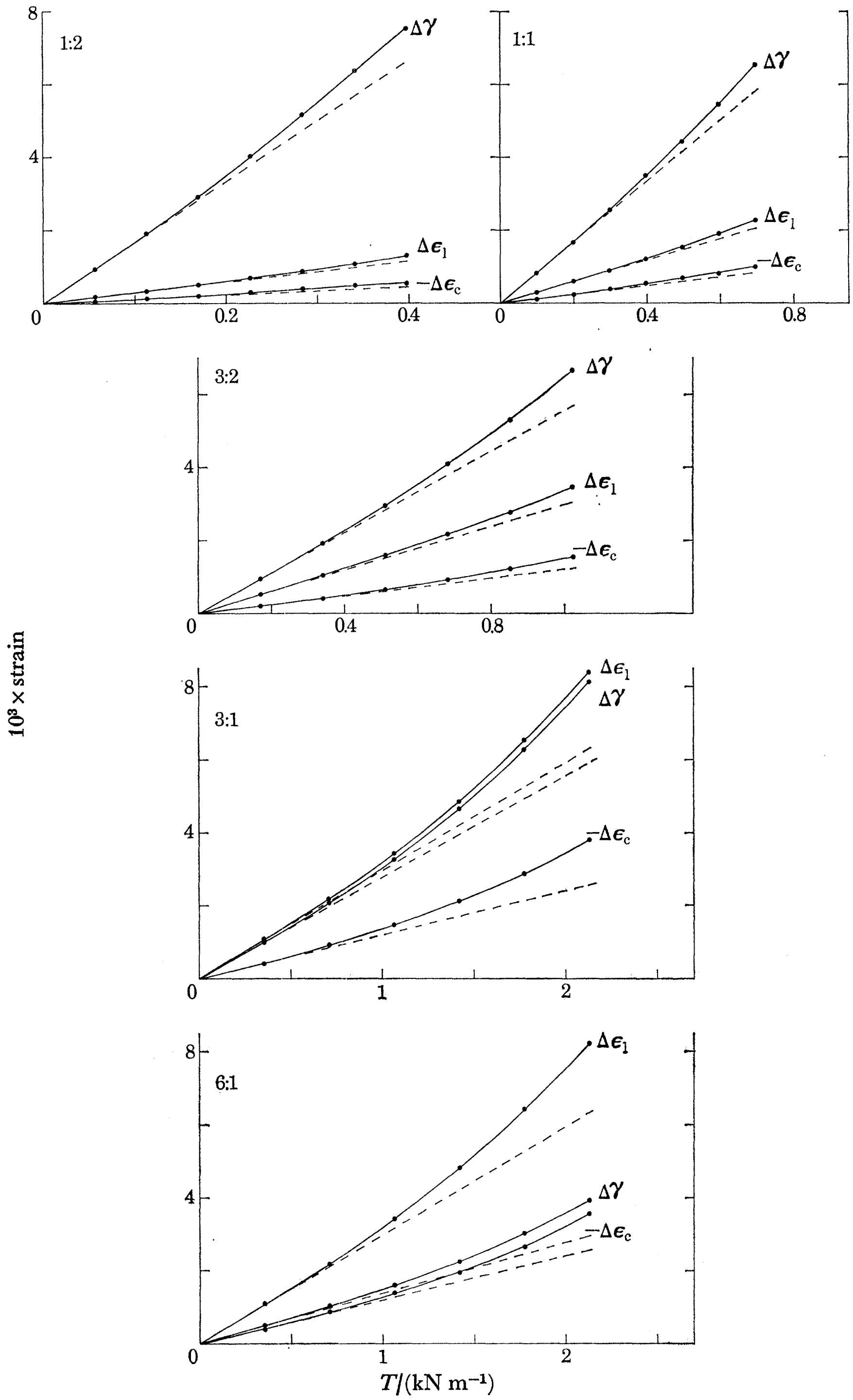

FIGURE 5. Recovery from creep under combined tension–torsion. Isochronal plots of recovery strain components against stress for $t-t_c = 10$ s, $t_c = 120$ s and various values of the ratio $T{:}S$. Circumferential recovery strain $\Delta\epsilon_c$ was calculated from $\Delta\epsilon_l$ and $\Delta\gamma$ using (4.15). Dashed lines show linear response.

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Now an empirical expression was recently proposed on intuitive grounds for describing creep of the same tube of polypropylene (Buckley & McCrum 1974). When expressed in two dimensional form it can be written

$$\boldsymbol{E}^{c}(t) = \frac{I_1}{w_0}\left[\frac{B}{9}-\frac{J}{6}\right]\boldsymbol{I}+\frac{J}{2w_0}\boldsymbol{P}, \tag{6.27}$$

where w_0 is the initial wall thickness. Also

$$B = B(t), \quad J = J(I_1, I_2, t), \tag{6.28}$$

and represent the three dimensional compressibility function and shear compliance function respectively. Comparing (6.25) with (6.27) we find that

$$\frac{\phi_0^c}{I_1} = \frac{1}{3w_0}\left[\frac{B}{3}-\frac{J}{2}\right], \quad \phi_1^c = \frac{J}{2w_0}, \tag{6.29}$$

from which it is clear that (6.27) and (6.28) imply the following relation between ϕ_0 and ϕ_1 for all I_1, I_2:

$$-\frac{\phi_0^c(I_1, I_2, t)}{I_1} = \frac{\phi_1^c(I_1, I_2, t)}{3} - \frac{B(t)}{9w_0}. \tag{6.30}$$

Since (6.30) must apply in the linear viscoelastic case of vanishingly small stresses it may be rewritten in the form

$$-\{(\phi_0^c/I_1)-\phi_0^{c*}\} = \tfrac{1}{3}(\phi_1^c-\phi_1^{c*}). \tag{6.31}$$

The results given in figure 3 were used to test the relation (6.31). Thus $-\phi_0^c/I_1$ and ϕ_1^c were calculated from $\epsilon_f^c(10\,\text{s})$ and $\gamma^c(10\,\text{s})$ using (4.15) and plotted one against the other in figure 6, for different values of the ratio $T:S$. Error bars indicate the possible random error from sources (*a*)–(*g*) mentioned above. Also shown in figure 6 is the straight line relation of slope $\frac{1}{3}$ predicted by (6.31). It is clear that, to within the precision of measurement, there is close agreement with (6.31), as would be expected from the agreement with (6.27) noted previously† (Buckley & McCrum 1974).

A similar check is possible for recovery following creep. For the creep experiment (6.24), a relation analogous to (6.25) applies during recovery at time $t(t > t_c)$ for the strains $\boldsymbol{E}^{r}(t-t_c, t_c)$ (see 6.15). Hence the recovery strains defined by (6.18) may be written

$$\Delta\boldsymbol{E}(t-t_c, t_c) = \Delta\phi_0\boldsymbol{I}+\Delta\phi_1\boldsymbol{P}, \tag{6.32}$$

where

$$\left.\begin{aligned}\Delta\boldsymbol{E}(t-t_c, t_c) &= \boldsymbol{E}^{c}(t)-\boldsymbol{E}^{r}(t-t_c, t_c),\\ \Delta\phi_0 &= \Delta\phi_0(I_1, I_2, t-t_c, t_c)\\ &= \phi_0^c(I_1, I_2, t)-\phi_0^r(I_1, I_2, t-t_c, t_c),\\ \Delta\phi_1 &= \Delta\phi_1(I_1, I_2, t-t_c, t_c)\\ &= \phi_1^c(I_1, I_2, t)-\phi_1^r(I_1, I_2, t-t_c, t_c).\end{aligned}\right\} \tag{6.33}$$

The analogous relation to (6.27) is

$$\Delta\boldsymbol{E}(t-t_c, t_c) = \frac{I_1}{w_0}\left[\frac{\bar{B}}{9}-\frac{\Delta J}{6}\right]\boldsymbol{I}+\frac{\Delta J}{2w_0}\boldsymbol{P}, \tag{6.34}$$

where

$$\bar{B} = B(t-t_c), \quad \Delta J = \Delta J(I_1, I_2, t-t_c, t_c), \tag{6.35}$$

† The results presented before in support of (6.27) referred to a creep time $t = 100$ s, while figure 6 refers to $t = 10$ s. This underlines the fact that the accuracy of (6.31) appears to be independent of t.

which, compared with (6.32) and making use of (6.19), leads to the analogue of (6.31)

$$-\{(\Delta\phi_0/I_1)-\overline{\phi}_0^{c*}\} = \tfrac{1}{3}(\Delta\phi_1-\overline{\phi}_1^{c*}), \tag{6.36}$$

where $\overline{\phi}_0^{c*} = \phi_0^{c*}(t-t_c), \quad \overline{\phi}_1^{c*} = \phi_1^{c*}(t-t_c).$

The recovery results given in figure 5 were used to compute $-\Delta\phi_0/I_1$ and $\Delta\phi_1$ from (6.14), (6.15) and (6.33); these are plotted in figure 7, together with the straight line relation (6.36) for com-

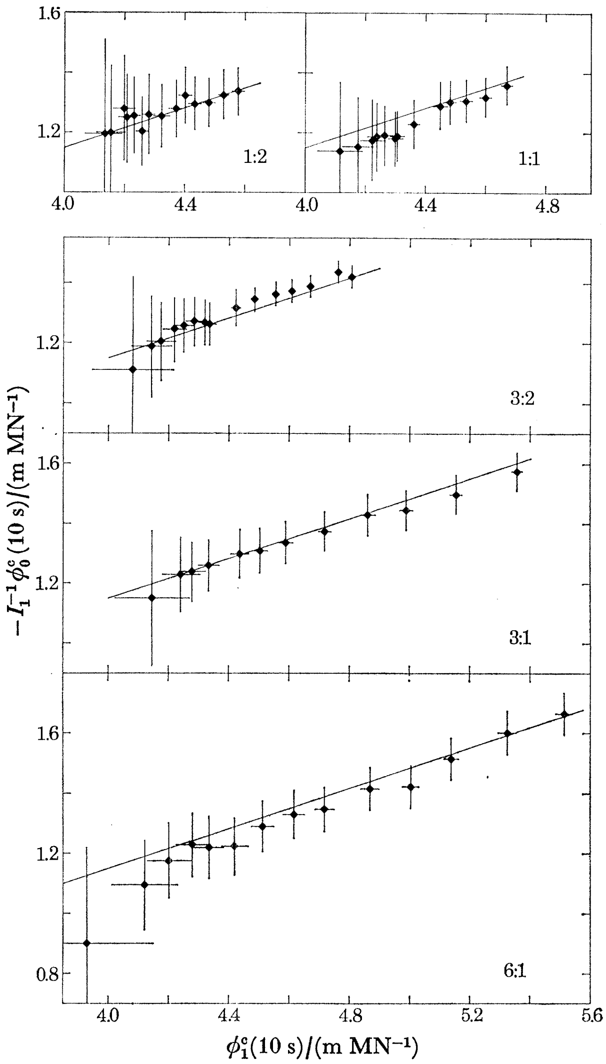

FIGURE 6. Experimental test of (6.31) during creep under combined tension–torsion. Isochronal plots of $-\phi^c(t)/I_1$ against $\phi_1^c(t)$ for t = 10 s and various values of the ratio T:S. Arms of the crosses indicate estimated limits of random error.

parison. Again, to within the precision of measurement, no systematic departure from (6.36) can be detected.

We thus conclude that, for our particular tube of polypropylene under combined tension–torsion, the response functions $-\phi_0/I_1$, ϕ_1 are linearly related through a gradient of $\frac{1}{3}$ for both creep and recovery experiments.

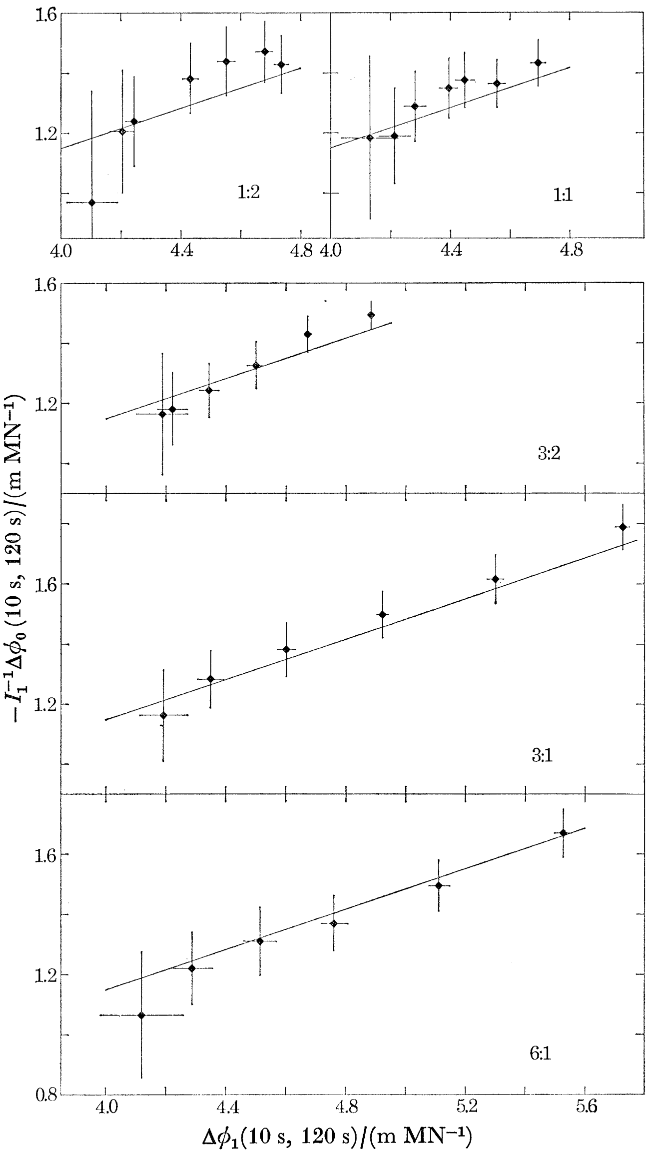

FIGURE 7. Experimental test of (6.36) during recovery from creep under combined tension–torsion. Isochronal plot of $\Delta\phi_0(t-t_c, t_c)/I_1$ against $\Delta\phi_1(t-t_c, t_c)$ for $t-t_c = 10$ s, $t_c = 120$ s and various values of the ratio $T{:}S$. Arms of the crosses indicate estimated limits of random error.

6.3. *Stress dependence of* ϕ_1

If ϕ_0 and ϕ_1 are indeed simply related, creep and recovery response of the tube is completely specified once the functional dependence of either one of these, say ϕ_1, on I_1 and I_2 is known under conditions of creep and recovery. We note, however, that we may equivalently consider ϕ_1 to depend on any two independent combinations of I_1 and I_2. From the point of view of gaining physical insight into the nature of the non-linearity and to aid comparison with the work of other authors, we prefer to consider the dependence of ϕ_1 on the more familiar invariants J_1, J_2' of the three dimensional stress tensor $\boldsymbol{\Sigma}$†

$$J_1 = \operatorname{tr}\boldsymbol{\Sigma}, \quad J_2' = \tfrac{1}{2}[\operatorname{tr}(\boldsymbol{\Sigma}')^2 - (\operatorname{tr}\boldsymbol{\Sigma}')^2], \tag{6.37}$$

† J_1, J_2' are related to the 'mean' stress Σ_m and 'effective' stress $\bar{\Sigma}$ widely used in classical plasticity through $\Sigma_m = \frac{1}{3}J_1$, $\bar{\Sigma} = \sqrt{(3J_2')}$.

where $\boldsymbol{\Sigma}'$ is the three dimensional deviatoric stress tensor defined by

$$\boldsymbol{\Sigma}' = \boldsymbol{\Sigma} - \tfrac{1}{3}\boldsymbol{I}\,\mathrm{tr}\,\boldsymbol{\Sigma}, \quad \mathrm{tr}\,\boldsymbol{\Sigma}' = 0. \tag{6.38}$$

When $\boldsymbol{\Sigma}$ reduces to two dimensional form, as it does in all circumstances within the scope of this paper, J_1, J_2' may be rewritten in terms of I_1, I_2:

$$w_0 J_1 = I_1, \quad w_0^2 J_2' = \tfrac{1}{2}[I_2 - \tfrac{1}{3}I_1^2]. \tag{6.39}$$

The significance of J_1, J_2' is that they separately characterize the hydrostatic and deviatoric components of the stress field, respectively. For creep and recovery under combined tension–torsion we have from (6.16) and (6.39)

$$w_0 J_1 = T, \quad w_0^2 J_2' = \tfrac{1}{3}T^2 + S^2. \tag{6.40}$$

The dependence of ϕ_1^c and $\Delta\phi_1$ on J_2', at constant J_1, was measured by using creep and recovery under pure torsion. This procedure was adopted to allow J_2' to approach zero. The loading programme employed took the form

$$S(\tau) = \sum_{n=1}^{m} nS_0 k_n(\tau), \quad T(\tau) = 0, \tag{6.41}$$

where on the nth pulse $J_1 = 0$, $w_0^2 J_2' = n^2 S_0^2$. As before, t_c and t_r were chosen to be 120 s and 1200 s and the loading could instead be considered to consist of isolated creep experiments of the form (6.5) (odd-numbered pulses) or

$$S(\tau) = S[h(\tau) - h(\tau - 2t_c)], \quad T(\tau) = 0, \tag{6.42}$$

(even-numbered pulses). Since (6.5) and (6.42) are special cases of (6.12), (6.22) with $T = 0$, ϕ_1^c and $\Delta\phi_1$ were calculated from γ^c and γ^r by using (6.14), (6.15) and (6.33). Isochronous results for $t = 10$ s and $t - t_c = 10$ s, $t_c = 120$ s respectively are shown in figure 8. Within the precision of measurement a linear relation with J_2' holds for both ϕ_1^c and $\Delta\phi_1$.

A loading programme was also devised which allowed ϕ_1^c and $\Delta\phi_1$ to be measured at regularly increasing values of J_1, while maintaining J_2' constant. This took the form

$$T(\tau) = \sum_{n=1}^{m} nT_0 j_n(\tau), \quad S(\tau) = \sum_{n=1}^{m} \sqrt{[w_0^2 J_2' - \tfrac{1}{3}n^2 T_0^2]}\, j_n(\tau), \tag{6.44}$$

where J_2' was held constant, while $w_0 J_1 = nT_0$ on the nth pulse. In this case t_c and t_r were chosen to be 200 s and 2000 s respectively and the loading could instead be considered to consist of isolated creep experiments of the form (6.12) with values of S and T chosen to maintain J_2', from (6.40), constant. To obtain $\phi_1^c(t)$ for t greater than t_c a 10 s linear extrapolation of the creep curve was employed in this instance. Again, ϕ_1^c and $\Delta\phi_1$ were calculated from γ^c and γ^r using (6.14), (6.15) and (6.33). They are plotted against J_1 in figure 9 for a constant value $w_0^2 J_2' = 0.729\,\mathrm{kN^2/m^2}$. This was chosen to allow the greatest range of J_1 to be covered while restricting all strains to be always less than 0.012. From the data of figure 9 it is clear that to within the precision of measurement both ϕ_1^c and $\Delta\phi_1$ are linear functions of J_1. There is also evidence that the difference $\Delta\phi_1 - \phi_1^c$ slightly *decreases* with increasing J_1, at this value of J_2', in contrast to its marked *increase* with increasing J_2' when $J_1 = 0$ (figure 8).

In order to observe the separate effects of J_1 and J_2' on the time dependence of ϕ_1^c, a few longer term creep experiments of type (6.12) were carried out, with $t_c = 10^4$ s, separated by times t_r of at least 10^5 s. The resulting time dependence of ϕ_1^c for different combinations of J_1 and J_2' is illustrated in figure 10. It is clear that the departure from linear viscoelastic behaviour with increasing J_1 at constant J_2' and with increasing J_2' at constant J_1 both become more marked with increasing creep time t.

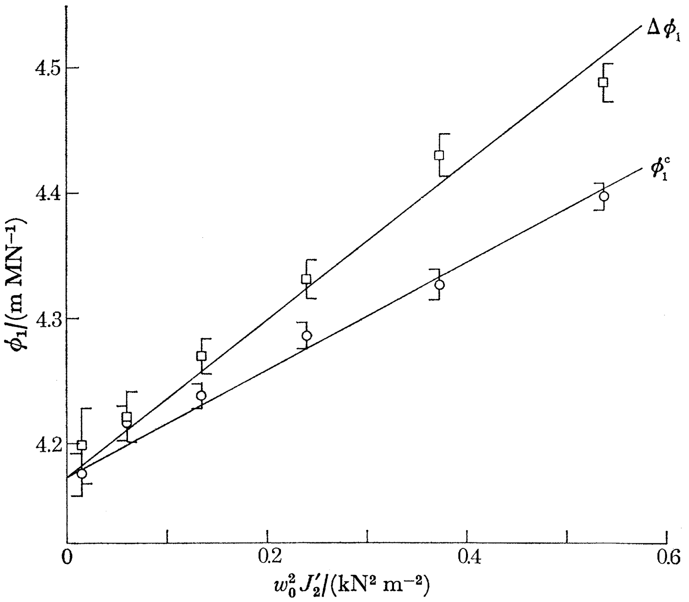

FIGURE 8. Dependence of ϕ_1 on invariant J_2' during creep and recovery under pure torsion ($J_1 = 0$). Isochronal data for $t = 10$ s and $t - t_c = 10$ s, $t_c = 120$ s respectively. Estimated limits of random error in ϕ_1 are shown.

6.4. *Further remarks*

For proportional loading in which the pressure difference p is zero, we have from (4.9) and (4.11)

$$\left.\begin{aligned} \epsilon_l - \epsilon_c &= \phi_1 P^{22}, \quad \gamma = 2\phi_1 P^{12}, \\ T(\tau) &= P^{22} f(\tau), \quad S(\tau) = P^{12} f(\tau). \end{aligned}\right\} \tag{6.45}$$

In the notation of § 6.3, ϕ_1 is a functional of $f(\tau)$ and a function of the invariants J_1, J_2' where

$$w_0 J_1 = P^{22}, \quad w_0^2 J_2' = \tfrac{1}{3}(P^{22})^2 + (P^{12})^2. \tag{6.46}$$

Now consider the separate responses to identical histories of pure axial tension, equal pure axial compression and pure torsion. These situations correspond respectively to

$$P^{22} = T, \qquad P^{12} = 0, \tag{6.47}$$

$$P^{22} = -T, \qquad P^{12} = 0, \tag{6.48}$$

$$P^{22} = 0, \qquad P^{12} = S. \tag{6.49}$$

The corresponding strain responses, at the same time t after the onset of $f(\tau)$ are, from (6.45) to (6.49),

$$\epsilon_l(T) - \epsilon_c(T) = T\phi_1(T, \tfrac{1}{3}T^2), \tag{6.50}$$

$$\epsilon_l(-T) - \epsilon_c(-T) = -T\phi_1(-T, \tfrac{1}{3}T^2), \tag{6.51}$$

$$\gamma(S) = 2S\phi_1(0, S^2), \tag{6.52}$$

where it is understood that in each case ϵ_l, ϵ_c, γ, ϕ_1 are all functionals of $f(\tau)$.

For the tube of polypropylene and the stress histories considered in § 6.3 the results recorded in figure 9 indicate that ϕ_1^c and $\Delta\phi_1$ are linear functions of J_1 for fixed J_2'. If there is a set of histories $f^*(\tau)$ for which the response function ϕ_1 in (6.50) to (6.52) is a linear function of J_1 for all values of J_2', it then follows that

$$3^{\frac{1}{2}}\gamma(S) = [\epsilon_l(T) - \epsilon_c(T)] - [\epsilon_l(-T) - \epsilon_c(-T)], \tag{6.53}$$

when we choose $T = 3^{\frac{1}{2}}S$ and $f = f^*$. The utility of (6.53) lies in its power to predict response to pure torsion from the results of experiments under pure axial tension and compression, for all histories $f^*(\tau)$. Although using different specimen geometries from that considered here, Benham (1973) found the correlation between tension, compression and shear given by (6.53) to hold with

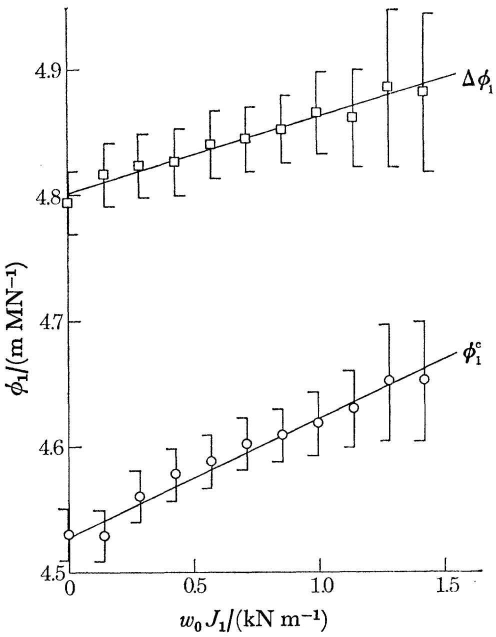

FIGURE 9

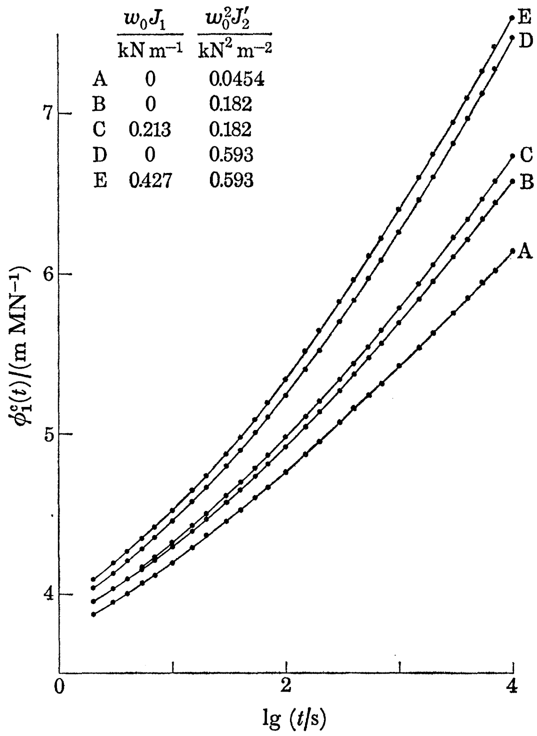

FIGURE 10

FIGURE 9. Dependence of ϕ_1 on invariant J_1 during creep and recovery under combined tension–torsion (values of S and T chosen to maintain constant $w_0^2 J_2' = 0.729\ \mathrm{kN^2/m^2}$). Isochronal data for $t = 10$ s and $t - t_c = 10$ s, $t_c = 200$ s respectively. Estimated limits of random error in ϕ_1 are shown.

FIGURE 10. Effect of invariants J_1 and J_2' on time dependence of $\phi_1^c(t)$ during creep under combined tension–torsion.

a remarkably high degree of precision for non-linear viscoelastic creep in polyvinyl chloride, polymethyl methacrylate and polypropylene. It may be observed that the result (6.53) also follows under the less restrictive assumption that, for a set of histories $f^*(\tau)$,

$$\left.\begin{aligned} \phi_1(J_1, J_2') &= \phi_1(0, J_2') + \overline{\phi}_1(J_1, J_2'), \\ \text{where} \qquad \overline{\phi}_1(J_1, J_2') &= -\overline{\phi}_1(-J_1, J_2'). \end{aligned}\right\} \tag{6.54}$$

Now consider another set of histories $f(\tau)$, which we denote by $f^+(\tau)$, corresponding to proportional loading, for which relations of the form (6.31) and (6.36) exist between ϕ_0 and ϕ_1. Thus, more generally,

$$-\{\phi_0/I_1 - \phi_0^*\} = \tfrac{1}{3}\{\phi_1 - \phi_1^*\}, \tag{6.55}$$

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where ϕ_0^*, ϕ_1^* are defined analogously to ϕ_0^{c*} and ϕ_1^{c*} in (6.9). As mentioned above, the stress dependence of ϕ_0 is then completely defined by that of ϕ_1. This implies a considerable saving of experimental effort in experiments involving proportional loading. From (4.11) and (6.55) determination of the complete strain response to a history $f^+(\tau)$, as a function of stress magnitude requires only (a) measurement of any *two* of ϵ_l, ϵ_c, γ in an experiment with vanishingly small I_1 and I_2 (in order to obtain ϕ_0^*, ϕ_1^*), and (b) measurement of any *one* of ϵ_l, ϵ_c, γ in experiments with different magnitudes of I_1 and I_2 in order to obtain, with the help of (6.55), both $\phi_0(I_1, I_2)$ and $\phi_1(I_1, I_2)$. As an example of the application of (6.55), consider the experiments of equal axial tension and compression, given by (6.47) and (6.48). From (4.15), (6.40), (6.47), (6.48) and (6.55) we find, at the same time t after the onset of f^+,

$$\left.\begin{aligned} \epsilon_l(T) &= T\{\phi_0^* + \tfrac{1}{3}\phi_1^* + \tfrac{2}{3}\phi_1(J_1, J_2')\},\\ \epsilon_c(T) &= T\{\phi_0^* + \tfrac{1}{3}\phi_1^* - \tfrac{1}{3}\phi_1(J_1, J_2')\},\\ w_0 J_1 &= T, \quad w_0^2 J_2' = \tfrac{1}{3}T^2, \end{aligned}\right\} \tag{6.56}$$

for axial tension, and

$$\left.\begin{aligned} \epsilon_l(-T) &= -T\{\phi_0^* + \tfrac{1}{3}\phi_1^* + \tfrac{2}{3}\phi_1(J_1, J_2')\},\\ \epsilon_c(-T) &= -T\{\phi_0^* + \tfrac{1}{3}\phi_1^* - \tfrac{1}{3}\phi_1(J_1, J_2')\},\\ w_0 J_1 &= -T, \quad w_0^2 J_2' = \tfrac{1}{3}T^2, \end{aligned}\right\} \tag{6.57}$$

for axial compression. On physical grounds, however, it might be expected that

$$\phi_1(J_1, J_2') > \phi_1(0, J_2') > \phi_1(-J_1, J_2') \tag{6.58}$$

when $J_1 > 0$. If this is the case it is clear from (6.56) and (6.57) that the material will be stiffer in compression than in tension, i.e.

$$\epsilon_l(T) > -\epsilon_l(-T), \quad -\epsilon_c(T) > \epsilon_c(-T). \tag{6.59}$$

The first inequality of (6.59) has been shown to hold, using other specimen geometries, during tensile and compressive creep of polyvinyl chloride, polymethyl methacrylate, polypropylene and polyacetal copolymer (Thomas 1969; Benham & Hutchinson 1971; Mallon, McCammond & Benham 1972). The second of (6.59) concerns the strains normal to the axis of tension or compression. It has been shown to hold by Mallon *et al.* (1972), again using specimen geometries other than that considered here, during tensile and compressive creep of polyvinyl chloride, polymethyl methacrylate and polypropylene. The relation (6.58) is consistent with the increase of ϕ_1^c and $\Delta\phi_1$ with J_1 found for polypropylene in the present work, and also with the fact that applying hydrostatic pressure ($J_1 < 0$) always decreases the strain response to a given applied stress, for a wide range of polymers (Ainbinder, Laka & Maiors 1965; Pugh, Chandler, Holliday & Mann 1971; Jones Parry & Tabor 1974; Sauer & Pae 1974).

We note, finally, that for histories f which belong to both sets f^* and f^+ a further economy of experimental effort is possible. The strain responses to axial compression may be predicted from the longitudinal strain $\epsilon_l(T)$ resulting from equal axial tension, if the shear response function $\phi_1(0, J_2')$ and linear viscoelastic responses ϕ_0^*, ϕ_1^* corresponding to f are known. Thus, combining (6.54), (6.56) and (6.57) we find

$$\left.\begin{aligned} \epsilon_l(-T) &= \epsilon_l(T) - 2T\{\phi_0^* + \tfrac{1}{3}\phi_1^* + \tfrac{2}{3}\phi_1(0, J_2')\},\\ \epsilon_c(-T) &= -\tfrac{1}{2}\epsilon_l(T) - \tfrac{1}{2}T\{\phi_0^* + \tfrac{1}{3}\phi_1^* - \tfrac{4}{3}\phi_1(0, J_2')\}. \end{aligned}\right\} \tag{6.60}$$

7. Conclusions

The foregoing sections have shown the present theory to be attractive in combining two desirable (and often mutually exclusive) features. First, it has wide generality in making no restriction, other than isotropy, on the viscoelastic behaviour of the tube. Secondly, it takes a form which is simple enough for routine application to experimental data.

It is unfortunate that it may be applied only in the case of small deformations. This was shown in § 3 to result from the fact that the applied force, couple and pressure must at all times balance stress components referred to the chosen convected coordinates. In practice the limitation is not serious, since in engineering applications of viscoelastic materials it is usual for strains to be restricted on a routine basis to be less than or of the order of 0.01.

The most important outcome of the present theory is that four material response functions ϕ_0, ϕ_1, ϕ_2, ϕ_3 are sufficient to describe completely deformations of the tube under the conditions prescribed. It is natural, therefore, to express experimental data in terms of these functions. In fact, when the stress history takes the more restricted form of two superposed proportional loadings only three functions ϕ_0, ϕ_1, ϕ_2 are sufficient. This class of stress history includes all experiments that can be performed with the two most usual configurations of axial force and torque or axial force and pressure difference. In the yet more simple case of proportional loading only two functions ϕ_0 and ϕ_1 are sufficient. This is the class of stress histories which was used in the creep and recovery experiments under combined axial force and torque described in §§ 5 and 6.

Results of these experiments on polypropylene confirmed that non-linearity of material response is significant even when deformations of the tube are small enough to allow application of the present theory. In fact, departures from linear viscoelastic behaviour were easily resolved at strains as low as 0.001. A result of some practical importance was that the present theory allowed the circumferential tensile strain ϵ_c to be calculated from the much more easily measurable strains ϵ_l and γ, during both creep and recovery.

The same experiments showed that, for the particular tube being considered, $-\phi_0/I_1$ is linearly related to ϕ_1 through a gradient of $\frac{1}{3}$. This is a concise statement of the intuitive notion that nonlinear behaviour can be described simply by allowing the shear creep compliance J to depend on stress history. It has two far-reaching implications. First, the experimental effort required in measuring the response of such a tube to proportional loading is considerably reduced, since only ϕ_1 need be measured. Secondly, by focusing attention on J it will stimulate the search for the mechanism of nonlinear creep. Much is already known of the physics governing J, for several of the practically important viscoelastic materials.

Non-linear behaviour of this particular tube was shown to arise in creep and recovery through ϕ_1 increasing with both J_1 and J_2' (defined in (6.37)). In creep, this was found to be the case for all creep times between 2 and 10^4 s. The difference $\Delta\phi_1 - \phi_1^c$, however, a manifestation of 'load interaction' nonlinearity, was found to increase with J_2' but not with J_1. These results show that the degree of nonlinearity depends not only on the current hydrostatic and deviatoric components of the applied stress but also on their history.

We conclude by noting that, although motivation for this work sprang from the needs of a common experimental test method, the results obtained may have wider significance. Several practical situations exist where thin-walled circular cylindrical tubes, of essentially isotropic viscoelastic material, are subject to stress systems which may lie within the class considered in this paper. The fields of structural engineering and bio-engineering provide two examples of topical interest – the increasing use of extruded thermoplastic pipes and the normal functioning of human blood vessels.

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